

FINITE TYPE AND THE EFFECTIVE NULLSTELLENSATZ

GORDON HEIER

ABSTRACT. Improved local and global versions of the effective Nullstellensatz for ideal sheaves on non-singular complex varieties are obtained, based on a new invariant motivated by the notion of finite type from the theory of several complex variables. Two closely related curve selection theorems for curves with maximal adjusted tangency orders to a given ideal sheaf are established along the way, using normalized blow-ups and integral closures.

1. INTRODUCTION

The notion of *finite type* plays a crucial role in the $\bar{\partial}$ -Neumann problem in several complex variables. In this Introduction, we will provide a short survey aimed at readers unfamiliar with finite type, subelliptic estimates, and subelliptic multiplier ideal sheaves. The subsequent sections are not logically dependent on the analytic phenomena described. Section 2 contains the statement and proof of the curve selection theorem for finite type ideals from [HL05], which has become part of this article (see page 4). Section 3 makes the connection with the Nullstellensatz in the spirit of [EL99]. The new angle is that, based on the notion of finite type, a new invariant for an ideal sheaf is defined that is well suited for the effective Nullstellensatz.

1.1. Finite type domains and subelliptic estimates. Let $\Omega \subset \mathbb{C}^n$ be a weakly pseudoconvex domain. We assume that Ω is given as $\{r < 0\}$ for some real-valued C^∞ function r whose gradient does not vanish anywhere on $\{r = 0\}$.

Kohn and Nirenberg ([KN65]) showed that local regularity for the $\bar{\partial}$ -Neumann problem at any point $p \in \bar{\Omega}$ is a consequence of a subelliptic a priori estimate, defined as follows. As usual, we restrict ourselves to the case of $(0, 1)$ -forms.

Definition 1.1. The $\bar{\partial}$ -Neumann problem for $(0, 1)$ -forms is said to satisfy a *subelliptic estimate (of order ε)* at p if there exists a neighborhood U of p , a constant $\varepsilon > 0$ and a constant $C > 0$ such that

$$\|\varphi\|_\varepsilon^2 \leq C(\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2 + \|\varphi\|^2)$$

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for all $\varphi \in \mathcal{D}_U^{0,1}$. Here, $\mathcal{D}_U^{0,1}$ denotes the space of $(0,1)$ -forms $\varphi \in \text{Dom}(\bar{\partial}^*)$ such that the coefficients of φ are C^∞ with compact support in $U \cap \bar{\Omega}$. The norm $\|\cdot\|_\varepsilon$ is the Sobolev ε -norm, taken coefficient-wise.

A subelliptic estimate of order 1 always holds for forms that are compactly supported in the interior of Ω , i.e. at $p \in \Omega$. For a boundary point $p_0 \in \partial\Omega$, a subelliptic estimate of order $\frac{1}{2}$ is known to hold if p_0 is a strongly pseudoconvex boundary point ([Koh63, Koh64]). It was proven by Catlin ([Cat83, Cat84, Cat87]) that a subelliptic estimate holds at p_0 if and only if p_0 is of *finite type* in the following sense. This notion was first introduced by D'Angelo ([D'A79]).

Definition 1.2. The *type* of $\partial\Omega$ at p_0 is defined to be

$$T(\partial\Omega, p_0) = \sup_{\gamma \in \Gamma_{p_0}} \frac{\text{ord}_0(r \circ \gamma)}{\text{ord}_0(\gamma)},$$

where Γ_{p_0} is the set of germs of non-constant local holomorphic arcs from $(\Delta, 0)$ to (\mathbb{C}^n, p_0) . We say that p_0 is a boundary point of *finite type* if $T(\partial\Omega, p_0) < \infty$. Geometrically, the type can be viewed as the maximal normalized touching order of holomorphic arcs with the boundary. It is independent of the choice of r .

In his series of papers, Catlin was even able to establish a *uniform effective* subelliptic estimate: he showed that a subelliptic estimate of order at least $t^{-n^2 t^{n^2}}$ holds at a boundary point p_0 with $T(\partial\Omega, p_0) = t$.

Before Catlin's work, Kohn ([Koh79]) had introduced an approach to subelliptic estimates using *subelliptic multiplier ideals*. The idea of multiplier ideals subsequently evolved into a fundamental concept of modern algebraic geometry, although a precise understanding of the relation between the different incarnations is still lacking. Note that Catlin's work uses appropriate families of exhaustion functions and does not make use of subelliptic multipliers. The relevant definitions are as follows.

Definition 1.3. For $p \in \bar{\Omega}$, let I_p be the set of all germs of complex-valued C^∞ functions F for which there exist a neighborhood U of p , $\varepsilon > 0$ and $C > 0$ (depending on F) such that

$$\|F\varphi\|_\varepsilon^2 \leq C(\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2 + \|\varphi\|^2)$$

for all $\varphi \in \mathcal{D}_U^{0,1}$. Elements of I_p are called *subelliptic multipliers*.

Clearly, a subelliptic estimate holds at p if and only if $1 \in I_p$. The following facts about I_p are established in [Koh79, Theorem 1.21]. I_p is an ideal, called the *subelliptic multiplier ideal*. It is equal to its own real radical, meaning that, for $k \in \mathbb{N}$, $|F|^k \leq |G|$ with $G \in I_p$ implies $F \in I_p$. It always contains the function r and the determinant of the Levi form. Most interestingly, [Koh79, Theorem 1.21, (d)] gives a procedure of generating

new multipliers from old ones: if F_1, \dots, F_j ($1 \leq j \leq n-1$) are in I_p , then the coefficients of

$$\partial F_1 \wedge \dots \wedge \partial F_j \wedge \partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-1-j}$$

are also in I_p . We call this last property (*).

Kohn's result is that in the case of r being real-analytic, the finite type condition at a boundary point p_0 is sufficient (and necessary) for the procedure to give $1 \in I_{p_0}$. Kohn's proof is based on property (*), the real radical property, and an induction on the dimension of the zero-set of a certain increasing sequence of subideals in I_{p_0} .

The question to what extent Kohn's procedure can be used to obtain a *uniform effective* subelliptic estimate at a boundary point similar to (or better than) Catlin's is being investigated. When applied verbatim, the procedure was recently found not to be always effective by means of the example

$$r = \operatorname{Re}(z_3) + |z_1^3 + z_1 z_2^b|^2 + |z_2|^2$$

(b any positive integer) at the origin in \mathbb{C}^3 . The author hopes to return to this problem.

1.2. Finite type ideals. An interesting special (but, e.g., not necessarily geometrically convex) case of the above general situation is when $p_0 = 0 \in \mathbb{C}^{n+1}$ and r (around 0) is of the form

$$(1) \quad r = \operatorname{Re}(z_{n+1}) + \sum_{i=1}^N |h_i(z_1, \dots, z_n)|^2,$$

where $N \in \mathbb{N}$, and the h_i are germs of holomorphic functions vanishing at the origin in \mathbb{C}^n , i.e. $h_i \in \mathfrak{m}_{\mathbb{C}^n, 0} \subset \mathcal{O}_{\mathbb{C}^n, 0}$.

It is easy to see that $T(\partial\Omega, 0) < \infty$ if and only if the ideal $\mathfrak{a} = (h_1, \dots, h_N)$ is $\mathfrak{m}_{\mathbb{C}^n, 0}$ -primary, i.e. there is a $q \in \mathbb{N}$ such that $\mathfrak{m}_{\mathbb{C}^n, 0}^q \subseteq \mathfrak{a} \subseteq \mathfrak{m}_{\mathbb{C}^n, 0}$. This means that the h_i vanish jointly precisely at the origin.

Motivated by the above, one can make the following definition.

Definition 1.4 ([D'A82]). Let $\mathfrak{a} \subseteq \mathcal{O}_{\mathbb{C}^n, 0}$ be an ideal of germs of holomorphic functions at the origin of \mathbb{C}^n with $\mathfrak{m}_{\mathbb{C}^n, 0}^q \subseteq \mathfrak{a} \subseteq \mathfrak{m}_{\mathbb{C}^n, 0}$ for some positive integer q . The *type* of \mathfrak{a} is the (a priori real) number

$$T(\mathfrak{a}) := \sup_{\gamma \in \Gamma_0} \inf_{f \in \mathfrak{a}} \left\{ \frac{\operatorname{ord}_0(f \circ \gamma)}{\operatorname{ord}_0(\gamma)} \right\},$$

where Γ_0 is the set of all germs of non-constant local holomorphic arcs $\gamma : \Delta \rightarrow \mathbb{C}^n$ with $\gamma(0) = 0$.

Equivalently, one can write

$$(2) \quad T(\mathfrak{a}) = \sup_{\gamma \in \Gamma_0} \left\{ \frac{\operatorname{ord}_0(\gamma^* \mathfrak{a})}{\operatorname{ord}_0(\gamma^* \mathfrak{m})} \right\},$$

where for an ideal $\mathfrak{b} \subseteq \mathcal{O}_{\mathbb{C}^n,0}$, $\text{ord}_0(\gamma^*\mathfrak{b})$ denotes the least order of vanishing of the pull-backs $\gamma^*(f)$ of germs of functions $f \in \mathfrak{b}$. It is well-known (and we shall see below) that $T(\mathfrak{a})$ is actually a rational number, and that the supremum is actually a maximum.

Remark 1.5. When r is of the form (1), then clearly

$$T(\partial\Omega, 0) = 2T(\mathfrak{a}),$$

where $\mathfrak{a} = (h_1, \dots, h_N)$.

Recently there has been some interest in understanding geometrically the curves that compute $T(\mathfrak{a})$. For example, McNeal and Némethi ([MN05]) proved that if $n = 2$, then one can specify finitely many curves $\gamma_1, \dots, \gamma_t$ so that $T(\mathfrak{a})$ is given by the maximum over the γ_α of the ratio on the right in (2). More on the 2-dimensional case can be found in [FJ04]. Using ideas introduced by Teissier ([Tei77, Tei82]), Lazarsfeld and the author ([HL05]) proved the analogous Theorem 2.1 in all dimensions, with Proposition 2.2 being the key component of the proof. We subsequently learned that these results were already contained in the unpublished seminar notes [LJT74] of Lejeune-Jalabert and Teissier and therefore decided not to publish our preprint. For the convenience of the reader, and as a warm-up for the more general situation dealt with in connection with the Nullstellensatz in Section 3, the material from [HL05] is reproduced in Section 2.

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2. CURVE SELECTION FOR FINITE TYPE IDEALS

2.1. Statement of the curve selection theorem. As a matter of terminology, one says that an element $f \in \mathfrak{b} \subseteq \mathcal{O}_{\mathbb{C}^n,0}$ is *general* if it is a general \mathbb{C} -linear combination of a collection of generators of \mathfrak{b} .

Theorem 2.1 ([LJT74], [HL05]). *Let $\mathfrak{a} = (h_1, \dots, h_N) \subseteq \mathfrak{m}_{\mathbb{C}^n,0}$ be an $\mathfrak{m}_{\mathbb{C}^n,0}$ -primary ideal. Choose $n - 1$ general elements $f_1, \dots, f_{n-1} \in \mathfrak{a}$ and let*

$$C = \text{Zeroes}(f_1, \dots, f_{n-1})_{\text{red}}$$

be the germ at 0 of the reduced curve arising as the common zeroes of f_1, \dots, f_{n-1} . Consider the decomposition

$$C = C_1 \cup \dots \cup C_t$$

of C into local analytic irreducible components, and let

$$\gamma_\alpha : \Delta \rightarrow C_\alpha$$

be the normalization of C_α . Then

$$T(\mathfrak{a}) = \max_{\alpha=1, \dots, t} \left\{ \frac{\text{ord}_0(\gamma_\alpha^*\mathfrak{a})}{\text{ord}_0(\gamma_\alpha^*\mathfrak{m})} \right\}.$$

2.2. Proof of the curve selection theorem. Fix a small neighborhood $U \subset \mathbb{C}^n$ of the origin in which the given ideal is represented by an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_U$. Let

$$\mathrm{Bl}_{\mathfrak{a}}(U) \longrightarrow U$$

be the blowing up of \mathfrak{a} . Concretely, using the generators $h_1, \dots, h_N \in \mathcal{O}(U)$ of \mathfrak{a} , $\mathrm{Bl}_{\mathfrak{a}}(U)$ can be realized as the closure in $U \times \mathbb{P}^{N-1}$ of the graph of the meromorphic mapping $U \dashrightarrow \mathbb{P}^{N-1}$ defined by h_1, \dots, h_N .

Now let $X^+ \longrightarrow \mathrm{Bl}_{\mathfrak{a}}(U)$ be the normalization of this blowing up, and denote by $\nu : X^+ \longrightarrow U$ the natural map. Then $\mathfrak{a} \cdot \mathcal{O}_{X^+} = \mathcal{O}_{X^+}(-F)$ for some effective Cartier divisor F on X^+ . Write

$$F = \sum_{i=1}^s r_i E_i$$

for the corresponding Weil divisor, where E_i are irreducible divisors on X^+ . Thus $r_i = \mathrm{ord}_{E_i}(\mathfrak{a})$ is the vanishing order along E_i of the pull-back $\nu^*(f)$ of a general element $f \in \mathfrak{a}$. We also let $m_i = \mathrm{ord}_{E_i}(\mathfrak{m})$, \mathfrak{m} denoting the maximal ideal sheaf defining the origin $0 \in U$.

Proposition 2.2 ([LJT74], [HL05]). *With the above notation, one has*

$$(3) \quad T(\mathfrak{a}) = \max_{i=1, \dots, s} \left\{ \frac{r_i}{m_i} \right\}.$$

To prove the Proposition, we will make use of the notion of integral closure of ideals. The reader may consult [Laz04a, Section 9.6.A] for a geometrically-oriented overview, or [Tei77, Tei82] for a more detailed treatment. For our purposes, it is sufficient to define the *integral closure* of integer powers \mathfrak{a}^k of \mathfrak{a} to be

$$\overline{\mathfrak{a}^k} = \nu_* \mathcal{O}_{X^+}(-kF).$$

An important fact is the following valuative criterion for membership in the integral closure of an ideal ([Tei82, Corollaire 2, p. 328]).

Lemma 2.3. *Given an ideal $\mathfrak{b} \subseteq \mathcal{O}_U$ vanishing only at the origin, one has $f \in \overline{\mathfrak{b}}$ if and only if*

$$\mathrm{ord}_0(\gamma^* f) \geq \mathrm{ord}_0(\gamma^* \mathfrak{b})$$

for all curves $\gamma \in \Gamma_0$. □

In particular, the type of an ideal only depends on the integral closure, i.e. for $\mathfrak{b} \subseteq \mathcal{O}_{\mathbb{C}^n, 0}$, one has $T(\mathfrak{b}) = T(\overline{\mathfrak{b}})$. Furthermore, Lemma 2.3 yields the following statement, whose simple proof is left to the reader.

Lemma 2.4. *The type $T(\mathfrak{a})$ is the least rational number $t > 0$ such that*

$$\mathfrak{m}^{kt} \subseteq \overline{\mathfrak{a}^k}$$

for all sufficiently large and divisible integers $k \in \mathbb{N}$. □

Proof of Proposition 2.2. Let τ denote the maximum on the right hand side in equation (3). Then for all $i = 1, \dots, s$, we have $\tau m_i \geq r_i$, which implies that

$$\left(\sum_{i=1}^s k\tau m_i E_i \right) - kF$$

is an effective divisor on X^+ . Let k be such that $k\tau \in \mathbb{N}$. Then

$$\mathfrak{m}^{k\tau} \subseteq \nu_* \mathcal{O}_{X^+} \left(- \sum_{i=1}^s k\tau m_i E_i \right) \subseteq \nu_* \mathcal{O}_{X^+} (-kF) = \overline{\mathfrak{a}^k}.$$

It follows from Lemma 2.4 that $\tau \geq T(\mathfrak{a})$. On the other hand, Lemma 2.4 also gives $\mathfrak{m}^{kT(\mathfrak{a})} \subseteq \overline{\mathfrak{a}^k}$ for sufficiently large and divisible k . Therefore,

$$kT(\mathfrak{a}) \operatorname{ord}_{E_i}(\mathfrak{m}) \geq k \operatorname{ord}_{E_i}(\mathfrak{a})$$

for all $i = 1, \dots, s$. This implies $\tau \leq T(\mathfrak{a})$. \square

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Given $n-1$ general elements $f_1, \dots, f_{n-1} \in \mathfrak{a}$, we write

$$D_j = \operatorname{div}(f_j), \quad \nu^* D_j = F + D'_j,$$

so that D'_j is the proper transform of D_j . Note that the D'_j generate a basepoint-free subseries of $| -F |$. Therefore, by Bertini, the curve

$$\Lambda = D'_1 \cap \dots \cap D'_{n-1}$$

is smooth (but possibly reducible) and meets $E = E_1 \cup \dots \cup E_s$ transversely at smooth points of intersection of Λ and E . Given an irreducible component C_α of C , let Λ_α denote its inverse image in Λ , and denote by $\gamma_\alpha : \Delta \cong \Lambda_\alpha \rightarrow C_\alpha$ the natural map. The component Λ_α is a smooth irreducible arc that meets E transversely at one point lying on a single irreducible component of E , say $E_{i(\alpha)}$. One also has

$$\operatorname{ord}_0(\gamma_\alpha^* \mathfrak{a}) = r_{i(\alpha)}, \quad \operatorname{ord}_0(\gamma_\alpha^* \mathfrak{m}) = m_{i(\alpha)}.$$

In view of Proposition 2.2, we are done if we can show that for each $i = 1, \dots, s$, there is an α such that $i(\alpha) = i$.

Now given i , the number of intersections of E_i with Λ is

$$(4) \quad (E_i \cdot \Lambda) = (E_i \cdot (-F)^{n-1}).$$

But $\mathcal{O}_{X^+}(-F)$ is relatively ample for ν (Lemma 2.5), so for all i the intersection number in (4) is strictly positive, which concludes the proof. \square

For the sake of completeness, we include the key ampleness assertion from the proof as a final lemma in this section.

Lemma 2.5. *With the above notation, $\mathcal{O}_{X^+}(-F)$ is ample on any subvariety contained in the support of F .*

Proof. In fact, F is the pull-back under the finite mapping $X^+ \rightarrow \text{Bl}_{\mathfrak{a}}(U)$ of the exceptional divisor F_0 on $Y = \text{Bl}_{\mathfrak{a}}(U)$. As explained above, we may view Y as an analytic subvariety of $U \times \mathbb{P}^{r-1}$, and $\mathcal{O}_Y(-F_0)$ is the pull-back of the hyperplane line bundle on \mathbb{P}^{r-1} , and it is ample on F_0 . Since ampleness is preserved under pulling back by finite maps, the statement follows. \square

3. THE EFFECTIVE NULLSTELLENSATZ

In this section, we will generalize the ideas of the previous section to obtain, in a very transparent manner, improved local and global versions of the effective Nullstellensatz, using the Theorem of Briançon-Skoda [SB74]. Ein-Lazarsfeld ([EL99]) and Hickel ([Hic01]), among others, did previous work in this direction. In Subsection 3.2, a curve selection theorem analogous to Theorem 2.1 is proven.

3.1. Local effective Nullstellensatz. We now take $\mathfrak{a} \subseteq \mathfrak{m}_{\mathbb{C}^n,0}$ to be any ideal, i.e. we no longer require it to be $\mathfrak{m}_{\mathbb{C}^n,0}$ -primary. Again, we fix a small neighborhood $U \subset \mathbb{C}^n$ of the origin in which the given ideal is represented by an ideal subsheaf $\mathfrak{a} \subseteq \mathcal{O}_U$.

We consider the zero-locus

$$Z = \text{Zeroes}(\mathfrak{a})_{\text{red}} \subset U$$

of \mathfrak{a} , viewed as a reduced subvariety of U . Let I_Z be the ideal of Z . According to Rückert's Nullstellensatz, $I_Z = \sqrt{\mathfrak{a}}$. Equivalently, one can say that there exists a $\sigma \in \mathbb{N}$ with $I_Z^\sigma \subseteq \mathfrak{a}$. We now generalize the notion of the type of an \mathfrak{m} -primary ideal discussed above to the following invariant.

Definition 3.1. For \mathfrak{a} as above, we define the invariant

$$\mathcal{T}(\mathfrak{a}) = \sup_{\gamma \in \Gamma} \left\{ \frac{\text{ord}_0(\gamma^* \mathfrak{a})}{\text{ord}_0(\gamma^* I_Z)} \right\},$$

where Γ is the set of local holomorphic arcs from $(\Delta, 0)$ to U whose image intersects Z , but is not contained in it.

Theorem 3.2. $\mathcal{T}(\mathfrak{a})$ governs the effective Nullstellensatz in the following way.

$$(\sqrt{\mathfrak{a}})^{\lceil n\mathcal{T}(\mathfrak{a}) \rceil} = I_Z^{\lceil n\mathcal{T}(\mathfrak{a}) \rceil} \subseteq \mathfrak{a}.$$

To prove the Theorem, we note that, in analogy to Lemma 2.4, the valuative criterion gives us the following.

Lemma 3.3. $\mathcal{T}(\mathfrak{a})$ is the least rational number $t > 0$ such that

$$I_Z^{kt} \subseteq \overline{\mathfrak{a}^k}$$

for all sufficiently large and divisible integers $k \in \mathbb{N}$. Moreover, for any positive integer k , we have

$$I_Z^{\lceil kt \rceil} \subseteq \overline{\mathfrak{a}^k}. \quad \square$$

Proof of Theorem 3.2. Apply Lemma 3.3 with $k = n$ to get $I_Z^{[nT(\mathfrak{a})]} \subseteq \overline{\mathfrak{a}^n}$. However, the Theorem of Briançon-Skoda [SB74] states that $\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}$ and the Theorem is proven. \square

3.2. A curve selection theorem. In the present situation, we can reintroduce the notation from the previous section: Let $\text{Bl}_{\mathfrak{a}}(U) \rightarrow U$ be the blowing up of \mathfrak{a} and $X^+ \rightarrow \text{Bl}_{\mathfrak{a}}(U)$ the normalization of this blowing up. Denote by $\nu : X^+ \rightarrow U$ the natural map. Then again $\mathfrak{a} \cdot \mathcal{O}_{X^+} = \mathcal{O}_{X^+}(-F)$ for some effective Cartier divisor F on X^+ . Again, write $F = \sum_{i=1}^s r_i E_i$ for the corresponding Weil divisor, where E_i are irreducible divisors on X^+ . Thus $r_i = \text{ord}_{E_i}(\mathfrak{a})$ is the vanishing order along E_i of the pull-back $\nu^*(f)$ of a general element $f \in \mathfrak{a}$. We also let $m_i = \text{ord}_{E_i}(I_Z)$. In the same way in which Lemma 2.4 was used to prove Proposition 2.2, Lemma 3.3 can be used to establish the following Proposition.

Proposition 3.4.

$$(5) \quad \mathcal{T}(\mathfrak{a}) = \max_{i=1, \dots, s} \left\{ \frac{r_i}{m_i} \right\}. \quad \square$$

We would like to describe a finite set of curves that compute $\mathcal{T}(\mathfrak{a})$. This is done in the following Theorem.

Theorem 3.5. *Let d denote the dimension of Z (i.e. the maximal dimension of its irreducible components). Take $f_1, \dots, f_{n-1} \in \mathfrak{a}$ and $g_1, \dots, g_d \in \mathcal{O}_U$ to be general elements. Let*

$$\begin{aligned} C^0 &= \overline{\text{Zeroes}(f_1, \dots, f_{n-1})_{\text{red}} \cap (U \setminus Z)} \\ C^1 &= \overline{\text{Zeroes}(f_1, \dots, f_{n-2}, g_1)_{\text{red}} \cap (U \setminus Z)} \\ &\vdots \\ C^d &= \overline{\text{Zeroes}(f_1, \dots, f_{n-1-d}, g_1, \dots, g_d)_{\text{red}} \cap (U \setminus Z)} \end{aligned}$$

be d curves (possibly reducible or empty). For $\delta = 0, \dots, d$, let $p_1^\delta, \dots, p_{M_\delta}^\delta$ be the points of intersection of C^δ with Z . In small neighborhoods $U(p_{\mu_\delta}^\delta)$ of each $p_{\mu_\delta}^\delta$ ($\mu_\delta = 1, \dots, M_\delta$), we take an irreducible decomposition

$$U(p_{\mu_\delta}^\delta) \cap C^\delta = C_1^{\delta, \mu_\delta} \cup \dots \cup C_{t_{\delta, \mu_\delta}}^{\delta, \mu_\delta}.$$

For $\alpha_{\delta, \mu_\delta} = 1, \dots, t_{\delta, \mu_\delta}$, we take normalization maps

$$\gamma_{\alpha_{\delta, \mu_\delta}}^{\delta, \mu_\delta} : \Delta \rightarrow C_{\alpha_{\delta, \mu_\delta}}^{\delta, \mu_\delta}.$$

Then

$$\mathcal{T}(\mathfrak{a}) = \max_{\delta=0, \dots, d; \mu_\delta=1, \dots, M_\delta; \alpha_{\delta, \mu_\delta}=1, \dots, t_{\delta, \mu_\delta}} \left\{ \frac{\text{ord}_0((\gamma_{\alpha_{\delta, \mu_\delta}}^{\delta, \mu_\delta})^* \mathfrak{a})}{\text{ord}_0((\gamma_{\alpha_{\delta, \mu_\delta}}^{\delta, \mu_\delta})^* I_Z)} \right\}.$$

Proof. For $j = 1, \dots, n-1$, write

$$D_j = \text{div}(f_j), \quad \nu^* D_j = F + D'_j,$$

and for $\iota = 1, \dots, d$

$$G_\iota = \operatorname{div}(g_\iota), \quad \nu^* G_\iota = G'_\iota.$$

For $\delta = 0, \dots, d$, let

$$\Lambda^\delta = D'_1 \cap \dots \cap D'_{n-1-\delta} \cap G'_1 \cap \dots \cap G'_\delta.$$

By Bertini's Theorem, Λ^δ is a smooth (but possibly reducible) curve which meets $E = E_1 \cup \dots \cup E_s$ transversely at smooth points of intersection with E . Given an irreducible component $C_{\alpha_\delta, \mu_\delta}^{\delta, \mu_\delta}$, let $\Lambda_{\alpha_\delta, \mu_\delta}^{\delta, \mu_\delta}$ be the component of Λ^δ that maps to it. We denote the natural isomorphism by $\gamma_{\alpha_\delta, \mu_\delta}^{\delta, \mu_\delta} : \Delta \cong \Lambda_{\alpha_\delta, \mu_\delta}^{\delta, \mu_\delta} \rightarrow C_{\alpha_\delta, \mu_\delta}^{\delta, \mu_\delta}$.

If $\Lambda_{\alpha_\delta, \mu_\delta}^{\delta, \mu_\delta}$ meets E in the component $E_{i(\alpha_\delta, \mu_\delta)}$, then

$$\operatorname{ord}_0((\gamma_{\alpha_\delta, \mu_\delta}^{\delta, \mu_\delta})^* \mathfrak{a}) = r_{i(\alpha_\delta, \mu_\delta)}, \quad \operatorname{ord}_0((\gamma_{\alpha_\delta, \mu_\delta}^{\delta, \mu_\delta})^* I_Z) = m_{i(\alpha_\delta, \mu_\delta)}.$$

From now on, we work with an arbitrary but fixed δ . In view of Proposition 3.4, it suffices to show that for all E_i with $\dim \nu(E_i) = \delta$, there exists a $\Lambda_{\alpha_\delta, \mu_\delta}^{\delta, \mu_\delta}$ which meets E_i .

The intersection of $\nu(E_i)$ with $g_1 \dots, g_\delta$ is a (nonempty) set of isolated points, and we can concentrate on any one of them, say p . The intersection number of $E_i \cap \nu^{-1}(p)$ with Λ^δ is

$$(6) \quad (E_i \cdot \nu^{-1}(p)) \cdot (-F)^{n-1-\delta}.$$

In the same way as in Lemma 2.5, one can see that $\mathcal{O}_{X+}(-F)|_{E_i \cap \nu^{-1}(p)}$ is ample, so the intersection number in (6) is strictly positive. \square

Remark 3.6. Based merely on the definition of the curves C^δ , it is not obvious that they are always nonempty when they are needed to be nonempty. Note that the positivity of (6) proves this fact in passing.

3.3. Geometric effective Nullstellensatz. Let X be a non-singular projective variety. In this subsection, we take \mathfrak{a} to be an ideal subsheaf of \mathcal{O}_X . Let L be an ample divisor on X such that $\mathcal{O}_X(L) \otimes \mathfrak{a}$ is globally generated.

We extend the definitions of Z and $\mathcal{T}(\mathfrak{a})$ in the obvious way to the present global situation, simply by replacing U by X . Note that the inclusion $I_Z^{[n\mathcal{T}(\mathfrak{a})]} \subseteq \mathfrak{a}$ established in Theorem 3.2 continues to hold true (because it is verified locally). The purpose of this subsection is to establish the fact that $\mathcal{T}(\mathfrak{a})$ is bounded by the top self-intersection of L :

Theorem 3.7. *With the above notation,*

$$\mathcal{T}(\mathfrak{a}) \leq (L^n).$$

We immediately obtain

Corollary 3.8 ([EL99, p. 431]). *With the above notation,*

$$(\sqrt{\mathfrak{a}})^{n(L^n)} = I_Z^{n(L^n)} \subseteq \mathfrak{a}.$$

Proof of Theorem 3.7. Let $\text{Bl}_{\mathfrak{a}}(X) \rightarrow X$ be the blowing up of \mathfrak{a} and $X^+ \rightarrow \text{Bl}_{\mathfrak{a}}(X)$ the normalization of this blowing up. Denote by $\nu : X^+ \rightarrow X$ the natural map. Then again $\mathfrak{a} \cdot \mathcal{O}_{X^+} = \mathcal{O}_{X^+}(-F)$ for some effective Cartier divisor F on X^+ . Again, write $F = \sum_{i=1}^s r_i E_i$ for the corresponding Weil divisor, where E_i are irreducible divisors on X^+ . Thus $r_i = \text{ord}_{E_i}(\mathfrak{a})$ is the vanishing order along E_i of the pull-back $\nu^*(f)$ of a general element $f \in \mathfrak{a}$. We also let $m_i = \text{ord}_{E_i}(I_Z)$. For the same reason as before, $\mathcal{T}(\mathfrak{a}) = \max_{i=1, \dots, s} \left\{ \frac{r_i}{m_i} \right\}$.

The following estimate is based on the telescoping sum trick used in [EL99, Proposition 3.1]. Let M be a divisor on X^+ linearly equivalent to $\nu^*L - F$. Note that M is nef. We have

$$\begin{aligned}
(L^n) &= ((\nu^*L)^n) \\
&\geq ((\nu^*L)^n) - (M^n) \\
&= ((\nu^*L - M) \cdot \left(\sum_{\mu=0}^{n-1} (\nu^*L)^\mu \cdot M^{n-1-\mu} \right)) \\
&= (F \cdot \left(\sum_{\mu=0}^{n-1} (\nu^*L)^\mu \cdot M^{n-1-\mu} \right)) \\
&= \sum_{i=1}^s (r_i E_i \cdot \left(\sum_{\mu=0}^{n-1} (\nu^*L)^\mu \cdot M^{n-1-\mu} \right)) \\
&\geq \sum_{i=1}^s (r_i E_i \cdot (\nu^*L)^{\dim \nu(E_i)} \cdot M^{n-1-\dim \nu(E_i)}) \\
&= \sum_{i=1}^s (r_i E_i \cdot (\nu^*L)^{\dim \nu(E_i)} \cdot (\nu^*L - F)^{n-1-\dim \nu(E_i)}) \\
&\geq \sum_{i=1}^s (r_i E_i \cdot (\nu^*L)^{\dim \nu(E_i)} \cdot (-F)^{n-1-\dim \nu(E_i)}) \\
&\geq \sum_{i=1}^s r_i.
\end{aligned}$$

Note that for the last inequality we are again exploiting the ν -ampleness of $-F$, which makes the intersection number that is the coefficient of r_i strictly positive for all i . On the whole, we have

$$\mathcal{T}(\mathfrak{a}) = \max_{i=1, \dots, s} \left\{ \frac{r_i}{m_i} \right\} \leq \sum_{i=1}^s r_i \leq (L^n). \quad \square$$

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RUHR-UNIVERSITÄT BOCHUM, FAKULTÄT FÜR MATHEMATIK, 44780 BOCHUM, GERMANY

Current address: University of Michigan, Mathematics Department, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109, USA

E-mail address: `heier@math.harvard.edu`